Summary for an Introductory EM Course

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Contents

1 Electric Field

Examinable materials:

- understand Coulomb's Law and interaction between point charges,
- understand the basic nature of charge,
- understand the concept of electric fields and the Superposition Principle,
- calculate electric fields for simple combinations of point charges,
- calculate electric fields from simple continuous charge distributions,
- calculate the electric field from a plane of charge and parallel combinations of planes of charge.

1.1 Coulomb's Law

The Coulomb force between two charges of charge q_1 and q_2 distance r apart is given by:

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2} \tag{1}$$

$$\vec{F} = -\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{\boldsymbol{r}}_{12} \tag{2}$$

1.2 $\vec{E}$-Field

The following equation gives the force on test charge q_0 in an electric field:

$$\vec{F} = q_0\vec{E} \tag{3}$$

Electric field lines points away from positive charges and towards negative charged.

The electric field at point $\vec{r}$ away from a charge q is given by:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\boldsymbol{r}} \tag{4}$$

1.3 Electric Dipole

Two charges of equal magnitude but opposite signs separated by a fixed distance d is known as an **electric dipole**. The electric field at distance equidistant to both charges r is given by:

$$\vec{E} = -\frac{1}{4\pi\epsilon_0}\frac{qd}{r^3}\hat{\boldsymbol{u}} \tag{5}$$

where $\hat{\boldsymbol{u}}$ is the unit vector that points from the positive to the negative charge.
The equation can be simplified by defining the **electric dipole moment vector**:

$$\vec{p} = qd\hat{\boldsymbol{u}} \tag{6}$$

which has units of coulomb meter. Equation 5 can be rewritten in terms of $\vec{p}$:

$$\vec{E} = -\frac{1}{4\pi\epsilon_0}\frac{\vec{p}}{r^3} \tag{7}$$

Note the $1/r^3$ dependence, this is characteristic of $\vec{E}$-field resulting from a dipole.

1.4 Charged Mass in Magnetic Field

Force experienced by a moving charge with velocity $\vec{v}$ and charge q through a magnetic field $\vec{B}$

1.5 Continuous Charge Distributions

Suppose we have a continuous charge distribution, the electric field can be found by applying the superposition principle:

$$\vec{E} = \sum_j \vec{E}_j = \int_{\text{all charges}} \mathrm{d}\vec{E} \tag{8}$$

Electric field on the axis of a ring can be found assuming a linear charge density:

$$\lambda = \frac{q}{2\pi R} \tag{9}$$

where q is the total charge on the ring and R is its radius. A infinitesimal charge element can be found considering a infinitesimal displacement element on the ring:

$$\mathrm{d}q = \lambda \mathrm{d}s \tag{10}$$

Integrating over the circumference of the ring will give the resultant electric field

$$E_z = \oint \frac{1}{4\pi\epsilon_0}\frac{z\lambda}{(R^2+z^2)^{3/2}}\mathrm{d}s \tag{11}$$

where z is the distance to the centre of the ring.

$$\vec{E}(z) = \frac{qz}{4\pi\epsilon_0(R^2+z^2)^{3/2}} \tag{12}$$

The **electric field from a charged disc** along the z-axis is:

$$\vec{E}(z) = \frac{q}{2\pi\epsilon_0 R^2}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right)\hat{\boldsymbol{k}} \tag{13}$$

For an **infinite plane of charge** with charge distribution σ, the electric field lines are parallel, the electric field is given by:

$$\vec{E} = \frac{\sigma}{2\epsilon_0}\hat{\boldsymbol{k}} \tag{14}$$

and between two plates:

$$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{\boldsymbol{k}} \tag{15}$$

2 Gauß's Law

2.1 Electric Flux

The electric flux is the electric field strength passing normal through a surface. For a charge q in the centre of a spherical surface radius r, the electric flux is the field strength times the surface area:

$$\Phi_E = \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)(4\pi r^2) = \frac{q}{\epsilon_0} \tag{16}$$

Gauss's Law may be written in a general form:

$$\Phi_E = \oint_S \vec{E}\cdot \mathrm{d}\vec{A} = \frac{q}{\epsilon_0} \tag{17}$$

which is true for **any** closed Gaussian surface enclosing a charge q.

3 Electrostatic Potential

The electrostatic potential is defined as the work necessary to bring a positive test charge from infinite distance to a point in the electric field:

$$V = -\int_\infty^A \vec{E}\cdot \mathrm{d}\vec{r} \tag{18}$$

The electrostatic potential due to a point charge q at a distance r away from the charge is:

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \tag{19}$$

The potential at the surface of a charged sphere of radius R holding charge Q is given by:

$$V_R = \frac{1}{4\pi\epsilon_0}\frac{Q}{R} \tag{20}$$

The **potential due to a dipole** is given by:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{\boldsymbol{r}}}{r^2} \tag{21}$$

3.1 Potential due to Continuous Charge Distribution

The potential along the z-axis from a charged ring of radius R carrying a linear charge distribution λ is given by:

$$V(z) = \frac{\lambda R}{2\epsilon_0(R^2+z^2)^{1/2}} \tag{22}$$

3.2 Electrostatic Potential Energy

The potential energy between two charges q_1 and q_2 separated by r is given by:

$$U = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r} \tag{23}$$

3.3 Dipole in an External Field

A dipole in an external field will experience a torque, given by:

$$\vec{\tau} = \vec{p}\times\vec{E} \tag{24}$$

The potential energy of a dipole in an external field is given by:

$$U = -\vec{p}\cdot\vec{E} \tag{25}$$

4 Capacitance

The capacitance and charge are related by:

$$Q = CV \tag{26}$$

The **capacitance of a parallel plate capacitor** is given by:

$$C = \frac{\epsilon_0 A}{d} \tag{27}$$

The capacitance of a spherical capacitor of inner and outer radii a and b respectively is given by:

$$C = 4\pi\epsilon_0\frac{ab}{b-a} \tag{28}$$

The capacitance of an isolated conducting sphere of radius R is:

$$C = 4\pi\epsilon_0 R \tag{29}$$

The capacitance of a cylindrical capacitor of length L is:

$$C = 2\pi\epsilon_0\frac{L}{\ln(b/a)} \tag{30}$$

4.1 Dielectric Materials

For a parallel plate capacitor of area A and plate separation d,

$$C = \frac{\varkappa \epsilon_0 A}{d} \tag{31}$$

The electric field inside a parallel plate capacitor is given by:

$$E = \frac{\sigma}{\varkappa \epsilon_0} \tag{32}$$

4.2 Capacitors in Circuits

Capacitors connected in parallel gives rise to a total linear sum of their capacitance:

$$C_p = \sum_{i=1}^{N} C_i \tag{33}$$

Capacitors connected in series would give the total capacitance in the following form:

$$\frac{1}{C_s} = \sum_{i=1}^{N} \frac{1}{C_i} \tag{34}$$

4.3 Potential Energy in a Capacitor

Capacitor stores potential energy in the electric field, the potential energy stored in given by

$$U = \frac{1}{2} C V^2 \tag{35}$$

The energy density within the capacitor is given by:

$$u_E = \frac{1}{2} \epsilon_0 E^2 \tag{36}$$

4.4 Current and Resistance

The current is defined as the rate of flow of charge:

$$I = \frac{\mathrm{d}q}{\mathrm{d}t} \tag{37}$$

and consequently:

$$Q = \int_0^t I(t) \mathrm{d}t \tag{38}$$

A more general definition of current is the **current density**, which is defined as follows:

$$I = \int \vec{J} \cdot \mathrm{d}\vec{A} \tag{39}$$

when the area element $\mathrm{d}\vec{A}$ is parallel to $\vec{J}$, the following is true:

$$I = JA \tag{40}$$

4.4.1 Resistance

A more general description of resistance is given by the ratio of the electric field of strength E and the resultant current density J:

$$\rho = \frac{E}{J} \tag{41}$$

The resistance of a material is given by:

$$R = \rho \frac{L}{A} \tag{42}$$

where L and A is the length and cross-sectional area respectively.

4.5 RC Circuit

In a RC circuit, the voltage decays exponentially:

$$V(t) = V_0 \exp\left(-\frac{t}{RC}\right) \tag{43}$$

where V_0 is the original voltage.

5 Magnetic Field

$$\vec{F} = q\vec{v} \times \vec{B} \tag{44}$$

For a particle with mass m, charge q and velocity $\vec{v}$ through a constant magnetic field $\vec{B}$, radius of its circular path and the period of one full revolution is given by:

$$r = \frac{mv}{qB} \tag{45}$$

$$T = \frac{2\pi r}{v} = \frac{2\pi m}{qB} \tag{46}$$

The kinetic energy of a particle with mass m, charge q in a **cyclotron** with magnetic field B, its kinetic energy at distance R away from the centre is given by:

$$K = \frac{q^2 B^2 R^2}{2m} \tag{47}$$

$$v \ll c$$

5.1 Lorentz Force

Lorentz force is a force experienced by a charged particle in a mixture of both electric and magnetic fields

$$\vec{F} = \vec{F}_E + \vec{F}_B \tag{48}$$

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{49}$$

A charged particle experiencing Lorentz force that arises from **perpendicular $\vec{B}$ and $\vec{E}$ fields,**

$$v = \frac{E}{B} \tag{50}$$

5.2 Magnetic Force on a Current

The definition for current, and the rate of change of charge per unit time:

$$I = \frac{\mathrm{d}q}{\mathrm{d}t} \tag{51}$$
$$N = nAv_D$$
$$I = nqAv_D$$

where n is the number density and v_D is the drift velocity, and A is the area of the plane cut through the wire.

For a wire of length L carrying a current I in the direction $\hat{l}$ in uniform magnetic field $\vec{B}$, N is the number of electron in the wire, the force on the wire is given:

$$\vec{F} = -eNv_D\hat{l} \times \vec{B} \tag{52}$$
$$\vec{F} = I\vec{L} \times \vec{B}$$

Conductors will experience **Hall Effect** in an external $\vec{B}$ field. For a conductor of width d in constant field $\vec{B}$ carrying a current I, it experiences force given by

$$F_b = ev_DB \tag{53}$$

which in turn induces a $\vec{E}$ field across the conductor, balancing the forces yields:

$$eE = ev_DB \tag{54}$$

which gives rise to **Hall Voltage**:

$$V_H = dv_DB \tag{55}$$

This allows for the measurement of **carrier density**:

$$n = \frac{BI}{V_H le} \tag{56}$$

where l is the thickness of the conductor. Rearranging Equation 56 gives magnetic field if Hall Voltage is known:

$$B = \frac{neV_H l}{I} \tag{57}$$

5.3 Torque on Current Loops

For a current loop with area A, carrying current I makes an angle θ with a magnetic field B experiences a torque τ:

$$\tau = AIB\sin\theta \tag{58}$$

In vectorial notation, where $\hat{n}$ is the normal vector to the plane formed by the loop (use right hand curl rule):

$$\vec{\tau} = IA\hat{n} \times \vec{B} \tag{59}$$

5.4 Magnetic Dipole Moment μ

For a current loop of area A carrying current I, magnetic dipole moment is given by

$$\vec{\mu} = IA\hat{n} \tag{60}$$

which simplifies Equation 59:

$$\vec{\tau} = \vec{\mu} \times \vec{B} \tag{61}$$

5.5 Potential Energy in Current Loops

*If torque is zero, then the potential energy **must be zero.***

The potential energy of a loop in $\vec{B}$ field is given by:

$$U(\theta) = -AIB\cos\theta \tag{62}$$
$$U(\theta) = -\vec{\mu}\cdot\vec{B} \tag{63}$$

6 Ampère's Law

Examinable Materials:

1. state the Biot-Savart Law and understand its implications
2. apply the Biot-Savart Law in simple geometrical cases
3. calculate the force between two parallel wires and explain as related to the definition of the Amp
4. understand the basis of the loop model for the electron and the relation between angular momentum and magnetic dipole moment
5. state and apply Ampère's law in simple geometries as well as toroidal solenoids

6.1 Biot-Savart Law

The **Biot-Savart Law** describes a wire carrying a current I, a small length element $\mathrm{d}s$, we find a small magnetic field element $\mathrm{d}B$ at pointed by located $\vec{r}$ away:

$$\mathrm{d}B = \frac{\mu_0}{4\pi}\frac{I\mathrm{d}s\sin\theta}{r^2} \tag{64}$$

In vector form:

$$\mathrm{d}\vec{B} = \frac{\mu_0}{4\pi}\frac{I\mathrm{d}\vec{s}\times\hat{r}}{r^2} \tag{65}$$

By applying the *superposition principle*, the total $\vec{B}$ at point $\vec{r}$ away from the current-carrying wire with current element $\mathrm{d}\vec{s}$ can be found:

$$\vec{B} = \frac{\mu_0}{4\pi}\int I\frac{\mathrm{d}\vec{s}\times\hat{r}}{r^2} \tag{66}$$

A circular current loop of radius R carrying a current I in a clockwise direction, the magnetic field B at the centre of said loop is given by:

$$B = \frac{\mu_0}{4\pi}\int_0^{2\pi R}\frac{I}{R^2}\mathrm{d}s \tag{67}$$

$$B = \frac{\mu_0 I}{2R}$$

A straight wire carrying current I, the magnetic field at a radial distance R is given by:

$$B = \frac{\mu_0 I}{2\pi}\int_0^{\infty}\frac{\sin\theta}{r^2}\mathrm{d}s \tag{68}$$

$$B = \frac{\mu_0 I}{2\pi R} \tag{69}$$

Force between two parallel wires, separated by a distance d and each carry current I_a and I_b respectively, will experience an attractive force if the currents are parallel, or a repulsive force if they are antiparallel:

$$F = \frac{\mu_0}{2\pi}\frac{LI_aI_b}{d} \tag{70}$$

6.2 Magnetic Field on Axis of A Loop

To find the axial magnetic field at point P on the axis a distance z away from the centre of a current loop wit radius R carrying a current I, first consider an element of loop $\mathrm{d}\vec{s}$, where the vector from said element to P is $\vec{r}$. $\mathrm{d}\vec{s}\perp\vec{r}$. At point P, the element $\mathrm{d}\vec{B}$ will be orthogonal to the plane defined by $\mathrm{d}\vec{s}$ and $\vec{r}$:

$$\mathrm{d}B = \frac{\mu_0}{4\pi}\frac{I\mathrm{d}s}{r^2}$$

At P, this can be written as two orthogonal components,

$$\mathrm{d}B_z = \mathrm{d}B\cos\alpha \tag{71}$$

$$\mathrm{d}B_\perp = \mathrm{d}B\sin\alpha \tag{72}$$

where α is the angle $\vec{r}$ makes with the plane of the current loop. By considering elements $\mathrm{d}\vec{s}$ in diametrically opposite pairs, each pair the perpendicular $\mathrm{d}B_\perp$ components cancel, such that the net $\vec{B}$ field points along the z- axis. By trigonometry:

$$\cos\alpha = \frac{R}{r} = \frac{R}{\sqrt{R^2+z^2}} \tag{73}$$

The Axial magnetic field for a loope of radius R carrying a current I is

$$B(z) = \frac{\mu_0 IR^2}{2(R^2+z^2)^{3/2}} \tag{74}$$

By approximating large distance away from the loop $z \ll R$,

$$B(z) \approx \frac{\mu_0 IR^2}{2z^3} \tag{75}$$

This effectively gives rise to a magnetic dipole, **the axial magnetic field due to a magnetic moment $\vec{\mu}$ is**:

$$\vec{B} = \frac{\mu_0}{2\pi}\frac{\vec{\mu}}{z^3} \tag{76}$$

6.3 Loop Model for Electron Orbits

The magnetic dipole moment that arises from orbiting electrons is given by:

$$\vec{\mu} = -\frac{e}{2m}\vec{L} \tag{77}$$

Recall that L is quantized by the magnetic quantum number m_l:

$$L = m_l\frac{h}{2\pi} \tag{78}$$

This leads to the quantisation of the magnetic dipole:

$$\mu_z = -m_l\frac{eh}{4\pi m} \tag{79}$$

where the fraction is known as the Bohr Magneton (μ_B).

6.4 Ampère's Law

For any closed loop, the enclosed current I is given by:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I \tag{80}$$

6.4.1 Magnetic Field of A Wire

By considering an Amperian loop of radius R bigger than the radius a of the wire, the $\vec{B}$ field can be found by:

$$B = \frac{\mu_0 I}{2\pi R} \tag{81}$$

The magnetic field within the wire is

$$B = \frac{\mu_0 I r}{2\pi a^2} \tag{82}$$

where a is the radius of the wire and r is the distance from the centre of the wire to the point the magnetic field B is interested.

6.5 Solenoids and Toroids

For a solenoid of n turns, the **magnetic field inside the solenoid** is:

$$B = \mu_0 n I \tag{83}$$

For a toroid of N turns, the magnetic field inside the toroid at radius R from the centre is given by:

$$B = \frac{\mu_0}{2\pi} \frac{NI}{R} \tag{84}$$

7 Induction

Examinable materials:

1. linking of E and B fields
2. understand and apply Farady and Lenz's Laws
3. understand induction in terms of conservation of energy
4. understand the reformulation of Faraday's Law and Maxwell's Law of induction
5. understand the concept of inductance and its dynamics in solenoidal systems
6. understand energy stored in magnetic fields and its relations to inductance
7. qualitatively understand mutual induction between two coils

7.1 Faraday's Law

The magnetic flux through a surface element $d\vec{A}$ is given by:

$$d\Phi_B = \vec{B} \cdot d\vec{A} \tag{85}$$

The total magnetic flux passing through a surface S is

$$\Phi_B = \int_S \vec{B} \cdot d\vec{A} \tag{86}$$

The induced emf is given by the change of magnetic flux through the loop with N turns:

$$\varepsilon = -N \frac{d\Phi_B}{dt} \tag{87}$$

Definition. *Lenz's Law states that the induced current has a direction such that the magnetic field due to this current* ***opposes*** *the change in the magnetic field that gave rise to the current.*

7.2 Faraday's Law in Time-Varying $\vec{B}$-field

A time varying $\vec{B}$ field will experience a induced $\vec{E}$ field, such that Faraday's Law may be written in terms of the induced electric field,

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \tag{88}$$

7.3 Maxwell's Law of Induction

Similar to Faraday's Law where changing magnetic field induces an electric field, a changing electric field will induce a magnetic field.

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \tag{89}$$

Note that this integral is **not applied to a Gaussian surface**, it is the electric flux through an open loop.

7.4 Ampère-Maxwell Law

Combining Equation 80 with the equation above, the magnetic field round a closed loop C, enclosing current I_e is given by:

$$\oint_C \vec{B} \cdot d\vec{s} = \mu_0 I_e + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \tag{90}$$

7.5 Displacement Current

The displacement current is a **fictitious induced current**, which generates a magnetic field according to Equation 90:

$$I_d = \epsilon_0 \frac{\mathrm{d}\Phi_E}{\mathrm{d}t} \tag{91}$$

Between two parallel plates of area A, when a current is applied between the to plates, this gives rise to a induced current I_d, and the **displacement current density** in the region between the plates is given by:

$$\vec{J}_d = \epsilon_0 \frac{\mathrm{d}\vec{E}}{\mathrm{d}t} \tag{92}$$

7.6 Inductors and Inductance

Analogous to capacitors, which store energy in an electric field, an inductor stores energy in a magnetic field.
The inductance of a solenoid with N turns and cross-sectional area A, and length l is given by:

$$L = \frac{N\Phi_B}{I}$$
$$L = \frac{\mu_0 N^2 A}{l} \tag{93}$$

The unit of inductance is $T\,m^2 A^{-1}$, *or* ***Henry****.*

7.7 Self Inductance

When a current is applied to a coil, this induces a back emf, resisting the change in current. The induced emf **for a solenoid** is given by:

$$\varepsilon_L = -L\frac{\mathrm{d}I}{\mathrm{d}t} \tag{94}$$

7.8 Resistor-Inductor Circuits (RL)

For a resistor with resistance R connected in series with inductor with inductance L, when a voltage V is applied, the current I is a function of time:

$$I(t) = \frac{V}{R}\left(1 - \exp\left(-\frac{R}{L}t\right)\right) \tag{95}$$

The ratio between L and R is a time known as **inductive time constant**:

$$\tau_L = \frac{L}{R} \tag{96}$$

7.9 Inductors in Circuits

N inductors with inductance L_i connected in series will give a total inductance of:

$$L_T = \sum_{i=1}^{N} L_i \tag{97}$$

N inductors with inductance L_i connected in parallel will give a total inductance of:

$$\frac{1}{L_T} = \sum_{i=1}^{N} \frac{1}{L_i} \tag{98}$$

7.10 Energy Stored in Magnetic Field

To store energy in a solenoid, we must do work against the self-induced emf in order to establish a $\vec{B}$ field. This is stored as a potential energy:

$$P = \varepsilon I \tag{99}$$

The total energy stored in a magnetic field due to a current I in a inductor is:

$$U_B = \int_0^I \mathrm{d}U_B = \int_0^I LI\mathrm{d}I = \frac{1}{2}LI^2 \tag{100}$$

Note the similarity to capacitance. This shows inductance is the sensible counter-part to capacitance in magnetism.

The energy density stored in a inductor is given by:

$$u_B = \frac{B^2}{2\mu_0} \tag{101}$$

7.11 Mutual Induction

Two coils with N_1 and N_2 turns sharing a common axis, a current I_1 is applied to one of the coils, which give rise to $\vec{B}_1$. Assuming there exists Φ_B through the second coil as a result of $\vec{B}_1$, the mutual inductance of the second coil w.r.t. the first is given by:

$$M_{21} = \frac{N_2 \Phi_B}{I_1} \tag{102}$$

8 Magnetism in Materials

Examinable materials:

1. explain the background of diamagnetism and paramagnetism in materials
2. explain the origins of ferromagnetism and why this can result in large permanent magnetic fields
3. explain the basic features of the Earth's magnetic field and outline its possible origin

8.1 Diamagnetism

Diamagnetism is where atoms of material have no permanent magnetic dipole moment. When diamagnetic materials are placed in a magnetic field, it induces a weak magnetic dipole moment due to a distortion of the electron orbitals. This induced magnetic field opposes the external field.

8.2 Paramagnetism

Contrary to diamagnetism, paramagnetic materials have a permanent magnetic dipole moment, but the moments are random, such that with out the presence of an external magnetic field, they cancel to zero. When an external field is applied, this results in an always attractive force, and the magnitude of the paramagnetic field is given by:

$$M = C\frac{B}{T} \tag{103}$$

where C is Curie's Constant, which is material-specific and M is the strength of the paramagnetic dipole moment per unit volume.

8.3 Ferromagnetism

Ferromagnetism is a phenomenon due to exchange coupling between atoms' dipole moments. Ferromagnetism allows for the possibility of permanent magnets. The magnetic property is lost at high temperatures (Curie Temperature). Ferromagnetic materials can be magnetised by an external $\vec{B}$ field through a process called hysteresis.

9 AC Circuits

Examinable materials:

1. Understand oscillations in LC circuits,
2. Understand the LRC circuit and the comparison with the mechanical damped simple harmonic oscillator,
3. Understand the phase shift between voltage and current and the basis of reactance,
4. Understand the analysis of forced RL and RC circuits using complex E,
5. Understand the basic driven LRC circuit and its relation to the classical mechanical force damped oscillator.

9.1 Potential Energy, LC

The **total potential energy** stored in a LC circuit with capacitor of capacitance C holding charge q, connected to a inductor with inductance L with current i flowing through the inductor is given by:

$$U_T = U_E + U_B = \frac{1}{2}\frac{q^2}{C} + \frac{1}{2}Li^2 \tag{104}$$

This quantity is conserved when the system is isolated.

9.2 Oscillations of LC Circuits

Applying the conservation of energy to Equation 104, we obtain the ODE that describes undamped simple harmonic oscillator:

$$\frac{dU_T}{dt} = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = 0 \tag{105}$$

$$i = \frac{dq}{dt} \tag{106}$$

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0 \tag{107}$$

Th solution of ODE is given:

$$q(t) = Q\cos(\omega_0 t) \tag{108}$$

$$\frac{dq}{dt} = i = -Q\omega_0\sin(\omega_0 t) \tag{109}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{110}$$

Note that due to the inductive load, the current lags behind the charge as a function of time by $\pi/2$.

9.3 Stored Potential Energy in LC Circuits

The stored potential energies in a LC circuit due to $\vec{E}$ and $\vec{B}$ fields are given by:

$$U_E = \frac{Q^2}{4C}(\cos(2\omega_0 t) + 1) \tag{111}$$

$$U_B = \frac{Q^2}{4C}(1 - \cos(2\omega_0 t)) \tag{112}$$

9.4 Damped LRC Circuit

The differential equation of a LRC circuit in terms of charge is given by:

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{LC}1 = 0 \tag{113}$$

where the linear derivative's coefficient comes from Ohm's Law.

The solution of the above ODE is:

$$q = Q\exp\left(-\frac{R}{2L}t\right)\cos(\omega t) \tag{114}$$

$$\omega_0 = \sqrt{\omega_0^2 - \left(\frac{R}{2L}\right)^2} \tag{115}$$

$$i = -Q\exp\left(-\frac{R}{2L}t\right)\left(\frac{R}{2L}\cos(\omega t) + \omega\sin(\omega t)\right) \tag{116}$$

For lightly damped case, $\omega \approx \omega_0$

9.5 Stored Potential Energy in LRC Circuits

The energy stored in a **low loss LRC circuit** is given by:

$$U_E = \frac{Q^2}{4C}\exp(-\gamma t)(\cos(2\omega t) + 1) \tag{117}$$

$$U_B = \frac{Q^2}{4C}\exp(-\gamma t)(1 - \cos(2\omega t)) \tag{118}$$

$$\gamma = \frac{R}{L} \tag{119}$$

For a lightly damped system, the quality factor quantifies the amount of damping:

$$Q_f = \frac{1}{R}\sqrt{\frac{L}{C}} \tag{120}$$

9.6 AC Power Source

9.6.1 Resistive Load

The phase difference between current I and voltage V is 0.

9.6.2 Capacitive Load

A capacitor will create a phase difference of $-\frac{\pi}{2}$ between I and V, a quantity called capacitive reactance is defined:

$$X_C = \frac{1}{\omega_d C} \tag{121}$$

where ω_d is the frequency of the supply. The following relation is true:

$$V = I_C X_C \tag{122}$$

9.6.3 Inductive Load

For a inductive load, the voltage leads the current by $\frac{\pi}{2}$

$$X_L = \omega_d L \tag{123}$$

$$V_L = I_L X_L \tag{124}$$

9.7 AC Voltage applied to RC and RL Circuits

By comparison to Ohm's Law, the peak current for a RC circuit with a AC peak voltage of ε is given by

$$I = \frac{\varepsilon}{Z} \tag{125}$$

$$Z = \sqrt{R^2 + X_L^2} \tag{126}$$

where Z is the impedance of the circuit.

For a RL circuit connected in series driven by an AC power source of maximum voltage ε_m, the impedance is given by:

$$Z = \sqrt{R^2 + (\omega_d L)^2} \tag{127}$$

$$I = \frac{\varepsilon_m}{\sqrt{R^2 + (\omega_d L)^2}} \tag{128}$$

For the same circuit, the instantaneous current can be written as:

$$i(t) = I\exp(\iota(\omega_d t - \phi)) \tag{129}$$

where $\iota = \sqrt{-1}$, I is the peak current and ϕ is the phase of the current.

The phase of the current, relative to the supply voltage is:

$$\phi = \arctan\left(\frac{\omega_d L}{R}\right) \tag{130}$$

For a driven RC circuit, the impedance and peak current is given by:

$$Z = \sqrt{R^2 + \frac{1}{(\omega_d C)^2}} \tag{131}$$

$$I = \frac{\varepsilon_m}{\sqrt{R^2 + \frac{1}{(\omega_d C)^2}}} \tag{132}$$

The phase of the current is give by:

$$\phi = \arctan\left(-\frac{1}{R\omega_d C}\right) \tag{133}$$

9.8 Forced LRC Circuit

For a circuit where a resistor with resistance R, inductor with inductance L, and a capacitor with capacitance C are connected in series, driven by a AC power source of the following form:

$$\varepsilon = \varepsilon_m \exp(\iota \omega_d t) \tag{134}$$

All three components experience an instantaneous current:

$$i = I \exp(\iota(\omega_d t - \phi)) \tag{135}$$

Furthermore, according to Kirchoff's Voltage Law,

$$\varepsilon = v_R + v_C + v_L \tag{136}$$

where v_R, v_C, and v_L are the instantaneous voltages at the corresponding components. Implementing the known phase shifts of the components ($R \to 0, L \to \frac{\pi}{2}, C \to -\frac{\pi}{2}$):

$$v_R = V_R \exp(\iota \omega_d t) \tag{137}$$

$$v_C = V_C \exp\left(\iota\left(\omega_d t + \frac{\pi}{2}\right)\right) \tag{138}$$

$$= \iota V_C \exp(\iota \omega_d t)$$

$$v_L = -\iota V_L \exp(\iota \omega_d t) \tag{139}$$

The above equations can be rearranged to solve for maximum current I:

$$I = \frac{\varepsilon_m}{\sqrt{R^2 + (X_L - X_C)^2}} \tag{140}$$

where the denominator of the above equation is the **impedance of the LRC circuit** Z.

10 Maxwell's Equations

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0} \tag{141}$$

$$\oint \vec{B} \cdot d\vec{A} = 0 \tag{142}$$

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \tag{143}$$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \tag{144}$$

The equations, by order of appearance:

1. **Gauss's Law for Electric Fields**, gives the electric field of charges enclosed in a Gaussian surface.
2. **Gauss's Law for Magnetic Fields**, gives the magnetic flux through a Gaussian surface, which must be zero.
3. **Faraday's Law of Induction**, for a closed loop defined by elements $d\vec{s}$, the time varying magnetic field generates an electric field. The negative sign is an artefact of **Lenz's Law**
4. **Ampère-Maxwell's Law of Magnetic Fields**, in a closed loop defined by elements $d\vec{s}$, the total magnetic fields are generated by currents as well as time-varying electric fields. This law unites electricity and magnetism.

10.1 Maxwell's Equations in Differential Form

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{145}$$

$$\nabla \cdot \vec{B} = 0 \tag{146}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{147}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{148}$$